Exercise 7

Find the Laplace transform of the following expressions:

$$1 + xe^x$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\, dx.$$

Use the definition to find the answer.

$$\begin{aligned}
\mathcal{L}\{1 + xe^x\} &= \int_0^\infty e^{-sx}(1 + xe^x)\, dx \\
&= \int_0^\infty (e^{-sx} + xe^x e^{-sx})\, dx \\
&= \int_0^\infty e^{-sx}\, dx + \int_0^\infty xe^x e^{-sx}\, dx
\end{aligned}$$

To avoid using integration by parts, write the second integrand as a derivative with respect to s.

$$\begin{aligned}
&= \frac{1}{(-s)} e^{-sx}\bigg|_0^\infty + \int_0^\infty \left(-\frac{\partial}{\partial s}\right) e^x e^{-sx}\, dx \\
&= \frac{1}{s} - \frac{d}{ds}\int_0^\infty e^{(1-s)x}\, dx \\
&= \frac{1}{s} - \frac{d}{ds}\left[\frac{1}{1-s} e^{(1-s)x}\bigg|_0^\infty\right] \\
&= \frac{1}{s} - \frac{d}{ds}\left(\frac{1}{s-1}\right) \\
&= \frac{1}{s} - \left[-\frac{1}{(s-1)^2}\right]
\end{aligned}$$

Therefore,

$$\mathcal{L}\{1 + xe^x\} = \frac{1}{s} + \frac{1}{(s-1)^2}.$$