Exercise 14

Convert each of the following Volterra integral equation in 9–16 to an equivalent IVP:

$$u(x) = 2 + \sinh x + \int_0^x (x-t)^2 u(t)\,dt$$

Solution

Differentiate both sides with respect to x.

$$u'(x) = \cosh x + \frac{d}{dx}\int_0^x (x-t)^2 u(t)\,dt$$

Use the Leibniz rule to differentiate the integral.

$$\begin{aligned} &= \cosh x + \left[\int_0^x \frac{\partial}{\partial x}(x-t)^2 u(t)\,dt + (0)^2 u(x)\cdot 1 - (x)^2 u(0)\cdot 0\right] \\ &= \cosh x + \left[\int_0^x 2(x-t)u(t)\,dt\right] \\ &= \cosh x + 2\int_0^x (x-t)u(t)\,dt \end{aligned}$$

Differentiate both sides with respect to x again.

$$\begin{aligned} u''(x) &= \sinh x + 2\frac{d}{dx}\int_0^x (x-t)u(t)\,dt \\ &= \sinh x + 2\left[\int_0^x \frac{\partial}{\partial x}(x-t)u(t)\,dt + (0)u(x)\cdot 1 - (x)u(0)\cdot 0\right] \\ &= \sinh x + 2\int_0^x u(t)\,dt \end{aligned}$$

Differentiate both sides with respect to x again.

$$\begin{aligned} u'''(x) &= \cosh x + 2\frac{d}{dx}\int_0^x u(t)\,dt \\ &= \cosh x + 2u(x) \\ u''' - 2u &= \cosh x \end{aligned}$$

The initial conditions to this ODE are found by plugging in $x = 0$ into the original integral equation,

$$u(0) = 2 + \sinh 0 + \int_0^0 (0-t)^2 u(t)\,dt = 2,$$

and the formula for u',

$$u'(0) = \cosh 0 + 2\int_0^0 (0-t)u(t)\,dt = 1,$$

and the formula for u'',

$$u''(0) = \sinh 0 + 2\int_0^0 u(t)\,dt = 0.$$

Therefore, the equivalent IVP is

$$u''' - 2u = \cosh x,\ u(0) = 2,\ u'(0) = 1,\ u''(0) = 0.$$