Exercise 16

Convert each of the following Volterra integral equation in 9–16 to an equivalent IVP:

$$u(x) = 1 + e^x + \int_0^x (1 + x - t)^3 u(t)\, dt$$

Solution

Differentiate both sides with respect to x.

$$u'(x) = e^x + \frac{d}{dx}\int_0^x (1 + x - t)^3 u(t)\, dt$$

Use the Leibniz rule to differentiate the integral.

$$\begin{aligned} &= e^x + \left[\int_0^x \frac{\partial}{\partial x}(1 + x - t)^3 u(t)\, dt + (1)^3 u(x)\cdot 1 - (1+x)^3 u(0)\cdot 0\right] \\ &= e^x + \int_0^x 3(1 + x - t)^2 u(t)\, dt + u(x) \\ &= u(x) + e^x + 3\int_0^x (1 + x - t)^2 u(t)\, dt \end{aligned}$$

Differentiate both sides with respect to x again.

$$\begin{aligned} u''(x) &= u'(x) + e^x + 3\frac{d}{dx}\int_0^x (1 + x - t)^2 u(t)\, dt \\ &= u'(x) + e^x + 3\left[\int_0^x \frac{\partial}{\partial x}(1 + x - t)^2 u(t)\, dt + (1)^2 u(x)\cdot 1 - (1+x)^2 u(0)\cdot 0\right] \\ &= u'(x) + e^x + 3\int_0^x 2(1 + x - t) u(t)\, dt + 3u(x) \\ &= u'(x) + 3u(x) + e^x + 6\int_0^x (1 + x - t) u(t)\, dt \end{aligned}$$

Differentiate both sides with respect to x again.

$$\begin{aligned} u'''(x) &= u''(x) + 3u'(x) + e^x + 6\frac{d}{dx}\int_0^x (1 + x - t) u(t)\, dt \\ &= u''(x) + 3u'(x) + e^x + 6\left[\int_0^x \frac{\partial}{\partial x}(1 + x - t) u(t)\, dt + (1) u(x)\cdot 1 - (1+x) u(0)\cdot 0\right] \\ &= u''(x) + 3u'(x) + e^x + 6\int_0^x u(t)\, dt + 6u(x) \\ &= u''(x) + 3u'(x) + 6u(x) + e^x + 6\int_0^x u(t)\, dt \end{aligned}$$

Differentiate both sides with respect to x again.

$$\begin{aligned} u^{(\text{iv})}(x) &= u'''(x) + 3u''(x) + 6u'(x) + e^x + 6\frac{d}{dx}\int_0^x u(t)\, dt \\ &= u'''(x) + 3u''(x) + 6u'(x) + e^x + 6u(x) \end{aligned}$$

$$u^{(\text{iv})} - u''' - 3u'' - 6u' - 6u = e^x$$

The initial conditions to this ODE are found by plugging in $x = 0$ into the original integral equation,

$$u(0) = 1 + e^0 + \int_0^0 (1 + 0 - t)^3 u(t)\, dt = 2,$$

and the formula for u',

$$u'(0) = u(0) + e^0 + 3\int_0^0 (1 + 0 - t)^2 u(t)\, dt = 3,$$

and the formula for u'',

$$u''(0) = u'(0) + 3u(0) + e^0 + 6\int_0^0 (1 + 0 - t) u(t)\, dt = 10,$$

and the formula for u''',

$$u'''(0) = u''(0) + 3u'(0) + 6u(0) + e^0 + 6\int_0^0 u(t)\, dt = 32.$$

Therefore, the equivalent IVP is

$$u^{(\text{iv})} - u''' - 3u'' - 6u' - 6u = e^x,\ u(0) = 2,\ u'(0) = 3,\ u''(0) = 10,\ u'''(0) = 32.$$

[TYPO: The answer at the back of the book reads:

$$u^{\text{iv}}(x) - u'''(x) - 3u''(x) - 6u(x) = e^x,\ u(0) = 2,\ u'(0) = 3,\ u''(0) = 10,\ u''' = 32$$

Parentheses need to be placed around "iv" and (0) needs to be placed after u'''. Also, $-6u'(x)$ is missing from the ODE.]