Exercise 6

Convert each of the following IVPs in 1–8 to an equivalent Volterra integral equation:

$$y''' - 2y'' + y = x, \; y(0) = 1, \; y'(0) = 0, \; y''(0) = 1$$

Solution

Let

$$y'''(x) = u(x). \tag{1}$$

Integrate both sides from 0 to x.

$$\begin{aligned} \int_0^x y'''(t)\,dt &= \int_0^x u(t)\,dt \\ y''(x) - y''(0) &= \int_0^x u(t)\,dt \end{aligned}$$

Substitute $y''(0) = 1$ and bring it to the right side.

$$y''(x) = 1 + \int_0^x u(t)\,dt \tag{2}$$

Integrate both sides again from 0 to x.

$$\begin{aligned} \int_0^x y''(s)\,ds &= \int_0^x \left[1 + \int_0^s u(t)\,dt\right] ds \\ y'(x) - y'(0) &= x + \int_0^x \int_0^s u(t)\,dt\,ds \end{aligned}$$

Substitute $y'(0) = 0$.

$$y'(x) = x + \int_0^x \int_0^s u(t)\,dt\,ds$$

Use integration by parts to write the double integral as a single integral. Let

$$\begin{aligned} v &= \int_0^s u(t)\,dt & dw &= ds \\ dv &= u(s)\,ds & w &= s \end{aligned}$$

and use the formula $\int v\,dw = vw - \int w\,dv$.

$$\begin{aligned} y'(x) &= x + s\int_0^s u(t)\,dt\bigg|_0^x - \int_0^x su(s)\,ds \\ &= x + x\int_0^x u(t)\,dt - \int_0^x su(s)\,ds \\ &= x + x\int_0^x u(t)\,dt - \int_0^x tu(t)\,dt \\ &= x + \int_0^x (x-t)u(t)\,dt \end{aligned} \tag{3}$$

Integrate both sides again from 0 to x.

$$\int_0^x y'(r)\,dr = \int_0^x \left[r + \int_0^r (r-t)u(t)\,dt\right] dr$$
$$y(x) - y(0) = \frac{x^2}{2} + \int_0^x \int_0^r (r-t)u(t)\,dt\,dr$$

Substitute $y(0) = 1$ and bring it to the right side.

$$y(x) = 1 + \frac{x^2}{2} + \int_0^x \int_0^r (r-t)u(t)\,dt\,dr$$

In order to evaluate the double integral, switch the order of integration so that dr comes first.

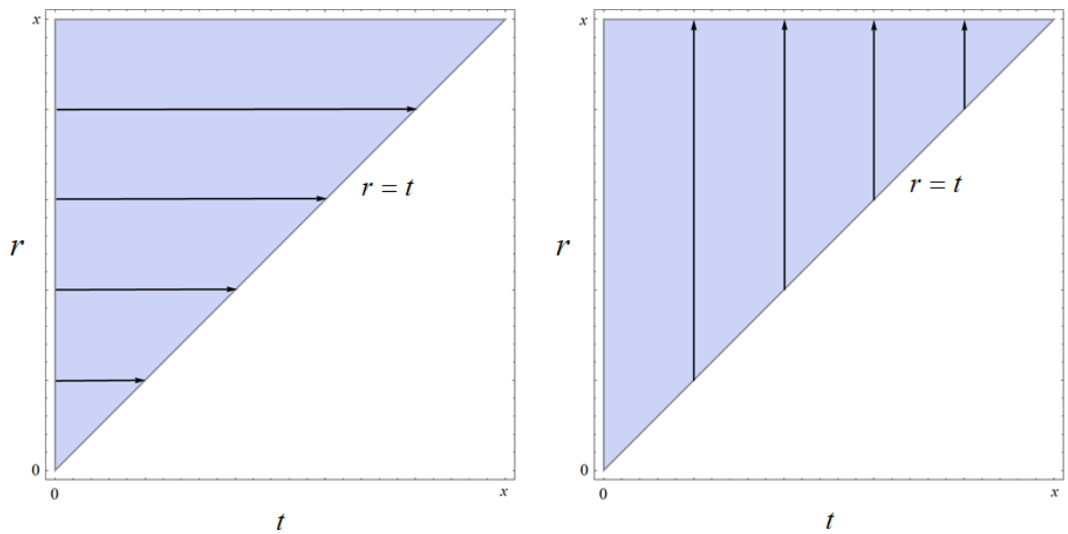

Figure 1: The current mode of integration in the tr-plane is shown on the left. This domain will be integrated over as shown on the right to simplify the integral.

$$\begin{aligned} y(x) &= 1 + \frac{x^2}{2} + \int_0^x \int_t^x (r-t)u(t)\,dr\,dt \\ &= 1 + \frac{x^2}{2} + \int_0^x \left[\frac{(r-t)^2}{2}\right]\Bigg|_t^x u(t)\,dt \\ &= 1 + \frac{x^2}{2} + \int_0^x \frac{(x-t)^2}{2} u(t)\,dt \\ &= 1 + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt \end{aligned} \tag{4}$$

Substitute equations (1), (2), (3), and (4) into the original ODE.

$$y''' - 2y'' + y = x \quad \to \quad u(x) - 2\left[1 + \int_0^x u(t)\,dt\right] + \left[1 + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt\right] = x$$

Expand the left side.

$$u(x) - 2 - 2\int_0^x u(t)\,dt + 1 + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt = x$$

$$u(x) - 1 + \frac{x^2}{2} + \int_0^x (-2)u(t)\,dt + \int_0^x \frac{1}{2}(x-t)^2 u(t)\,dt = x$$

$$u(x) - 1 + \frac{x^2}{2} + \int_0^x \left[-2 + \frac{1}{2}(x-t)^2\right] u(t)\,dt = x$$

$$u(x) = 1 + x - \frac{x^2}{2} - \int_0^x \left[-2 + \frac{1}{2}(x-t)^2\right] u(t)\,dt$$

Therefore, the equivalent Volterra integral equation is

$$u(x) = 1 + x - \frac{x^2}{2} + \int_0^x \left[2 - \frac{1}{2}(x-t)^2\right] u(t)\,dt.$$

This answer is in disagreement with the answer at the back of the book,

$$u(x) = 1 + x - \frac{1}{2}x^2 + 2\int_0^x \left[1 - \frac{1}{2}(x-t)^2\right] u(t)\,dt.$$

The general solution to the ODE, $y''' - 2y'' + y = x$, is

$$y(x) = C_1 e^{\frac{1}{2}(1-\sqrt{5})x} + C_2 e^{\frac{1}{2}(1+\sqrt{5})x} + C_3 e^x + x.$$

Using the initial conditions, $y(0) = 1$, $y'(0) = 0$, and $y''(0) = 1$, the constants of integration, C_1, C_2, and C_3, can be determined.

$$C_1 = 1 + \frac{\sqrt{5}}{5}$$

$$C_2 = \frac{2(3\sqrt{5} - 5)}{5(\sqrt{5} - 1)}$$

$$C_3 = -1$$

Plugging these in to $y(x)$ and then taking three derivatives of it gives us $u(x)$ by equation (1).

$$\begin{aligned} u(x) &= y'''(x) \\ &= \frac{1}{5}e^{\frac{1}{2}(1-\sqrt{5})x}\left[5 - 3\sqrt{5} + (5 + 3\sqrt{5})e^{\sqrt{5}x} - 5e^{\frac{1}{2}(1+\sqrt{5})x}\right] \end{aligned}$$

This solution satisfies the Volterra integral equation I obtained but not the one at the back of the book.