Exercise 13

Convert each of the following Fredholm integral equation in 9–16 to an equivalent BVP:

$$u(x) = e^{3x} + \int_0^1 K(x,t)u(t)\,dt,\ K(x,t) = \begin{cases} t, & \text{for } 0 \le t \le x \\ x, & \text{for } x \le t \le 1 \end{cases}$$

Solution

Substitute the given kernel $K(x,t)$ into the integral.

$$u(x) = e^{3x} + \int_0^x tu(t)\,dt + \int_x^1 xu(t)\,dt \tag{1}$$

Differentiate both sides with respect to x.

$$u'(x) = 3e^{3x} + \frac{d}{dx}\int_0^x tu(t)\,dt + \frac{d}{dx}\int_x^1 xu(t)\,dt$$

Apply the Leibnitz rule to differentiate the second integral.

$$\begin{aligned} &= 3e^{3x} + xu(x) + \int_x^1 \frac{\partial}{\partial x}xu(t)\,dt + xu(1)\cdot 0 - xu(x)\cdot 1 \\ &= 3e^{3x} + \int_x^1 u(t)\,dt \\ &= 3e^{3x} - \int_1^x u(t)\,dt \end{aligned} \tag{2}$$

Differentiate both sides with respect to x once more.

$$\begin{aligned} u''(x) &= 9e^{3x} - \frac{d}{dx}\int_1^x u(t)\,dt \\ &= 9e^{3x} - u(x) \end{aligned}$$

The boundary conditions are found by setting $x = 0$ and $x = 1$ in equations (1) and (2), respectively.

$$\begin{aligned} u(0) &= e^0 + \int_0^0 tu(t)\,dt + \int_0^1 (0)u(t)\,dt = 1 \\ u'(1) &= 3e^3 - \int_1^1 u(t)\,dt = 3e^3 \end{aligned}$$

Therefore, the equivalent BVP is

$$u'' + u = 9e^{3x},\ u(0) = 1,\ u'(1) = 3e^3.$$