Exercise 10

In Exercises 9–12, show that the given function $u(x)$ is a solution of the corresponding Fredholm integro-differential equation:

$$u'(x) = e^x + (e-1) - \int_0^1 u(t)\,dt,\ u(0) = 1,\ u(x) = e^x$$

Solution

Substitute the function in question on both sides of the integro-differential equation.

$$\begin{aligned}
\frac{d}{dx}(e^x) &\stackrel{?}{=} e^x + (e-1) - \int_0^1 e^t\,dt \\
e^x &\stackrel{?}{=} e^x + e - 1 - e^t\Big|_0^1 \\
&\stackrel{?}{=} e^x + e - 1 - (e^1 - e^0) \\
&\stackrel{?}{=} e^x + e - 1 - e + 1 \\
&= e^x
\end{aligned}$$

Therefore,

$$u(x) = e^x$$

is a solution of the Fredholm integro-differential equation.