Exercise 16

In Exercises 13–16, show that the given function $u(x)$ is a solution of the corresponding Volterra integro-differential equation:

$$u''(x) = 1 - xe^{-x} - \int_0^x tu(t)\,dt,\; u(0) = 1,\; u'(0) = -1,\; u(x) = e^{-x}$$

Solution

Substitute the function in question on both sides of the integro-differential equation.

$$\begin{aligned} \frac{d^2}{dx^2}(e^{-x}) &\stackrel{?}{=} 1 - xe^{-x} - \int_0^x te^{-t}\,dt \\ e^{-x} &\stackrel{?}{=} 1 - xe^{-x} - \int_0^x te^{-t}\,dt \end{aligned}$$

To solve the integral, introduce a new variable s in the exponent.

$$\begin{aligned} e^{-x} &\stackrel{?}{=} 1 - xe^{-x} - \int_0^x te^{-st}\,dt\bigg|_{s=1} \\ &\stackrel{?}{=} 1 - xe^{-x} + \int_0^x \frac{\partial}{\partial s}e^{-st}\,dt\bigg|_{s=1} \\ &\stackrel{?}{=} 1 - xe^{-x} + \frac{d}{ds}\int_0^x e^{-st}\,dt\bigg|_{s=1} \\ &\stackrel{?}{=} 1 - xe^{-x} + \frac{d}{ds}\left(-\frac{1}{s}e^{-st}\bigg|_0^x\right)\bigg|_{s=1} \\ &\stackrel{?}{=} 1 - xe^{-x} + \frac{d}{ds}\left[-\frac{1}{s}(e^{-sx} - 1)\right]\bigg|_{s=1} \\ &\stackrel{?}{=} 1 - xe^{-x} + \left[\frac{1}{s^2}(e^{-sx} - 1) - \frac{1}{s}(-xe^{-sx})\right]\bigg|_{s=1} \\ &\stackrel{?}{=} 1 - xe^{-x} + \left[(e^{-x} - 1) - (-xe^{-x})\right] \\ &\stackrel{?}{=} \cancel{1} - \cancel{xe^{-x}} + e^{-x} - \cancel{1} + \cancel{xe^{-x}} \\ &= e^{-x} \end{aligned}$$

Therefore,

$$u(x) = e^{-x}$$

is a solution of the Volterra integro-differential equation.