Exercise 22

In Exercises 17–24, find the unknown if the solution of each equation is given:

$$\text{If } u(x) = \sin x \text{ is a solution of } u(x) = f(x) + \frac{4}{\pi}\int_0^x \int_0^{\frac{\pi}{2}} u^2(t)\,dt\,dt, \text{ find } f(x)$$

Solution

Substitute the solution into both sides of the equation.

$$\begin{aligned}
\sin x &= f(x) + \frac{4}{\pi}\int_0^x \int_0^{\frac{\pi}{2}} \sin^2 t\,dt\,dt \\
&= f(x) + \frac{4}{\pi}\int_0^x \int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2t)\,dt\,dt \\
&= f(x) + \frac{2}{\pi}\int_0^x \int_0^{\frac{\pi}{2}} (1-\cos 2t)\,dt\,dt \\
&= f(x) + \frac{2}{\pi}\left(\int_0^x \int_0^{\frac{\pi}{2}} dt\,dt - \int_0^x \int_0^{\frac{\pi}{2}} \cos 2t\,dt\,dt\right) \\
&= f(x) + \frac{2}{\pi}\left(\frac{\pi}{2}x - \underbrace{\int_0^x \frac{1}{2}\sin 2t\Big|_0^{\frac{\pi}{2}}\,dt}_{=0}\right) \\
&= f(x) + x
\end{aligned}$$

Therefore,

$$f(x) = \sin x - x.$$