Exercise 6

In Exercises 5–8, show that the given function $u(x)$ is a solution of the corresponding Volterra integral equation:

$$u(x) = 4x + \sin x + 2x^2 - \cos x + 1 - \int_0^x u(t)\,dt, \ u(x) = 4x + \sin x$$

Solution

Substitute the function in question on both sides of the integral equation.

$$4x + \sin x \stackrel{?}{=} 4x + \sin x + 2x^2 - \cos x + 1 - \int_0^x (4t + \sin t)\,dt$$

Subtract $4x + \sin x$ from both sides and split the integral into two.

$$\begin{aligned}
0 &\stackrel{?}{=} 2x^2 - \cos x + 1 - \left(\int_0^x 4t\,dt + \int_0^x \sin t\,dt\right) \\
&\stackrel{?}{=} 2x^2 - \cos x + 1 - \left[2t^2\Big|_0^x + (-\cos t)\Big|_0^x\right] \\
&\stackrel{?}{=} 2x^2 - \cos x + 1 - \left[2x^2 - 0 + (-\cos x) - (-\cos 0)\right] \\
&\stackrel{?}{=} 2x^2 - \cos x + 1 - 2x^2 + \cos x - 1 \\
&= 0
\end{aligned}$$

Therefore,

$$u(x) = e^{2x}$$

is a solution of the Volterra integral equation.