Exercise 1

In Exercises 1–26, solve the following Volterra integral equations by using the *Adomian decomposition method*:

$$u(x) = 6x - 3x^2 + \int_0^x u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= 6x - 3x^2 + \int_0^x \sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= 6x - 3x^2 + \int_0^x [u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{6x - 3x^2}_{u_0(x)} + \underbrace{\int_0^x u_0(t)\,dt}_{u_1(x)} + \underbrace{\int_0^x u_1(t)\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. After enough terms are written, a pattern can be noticed, allowing us to write a general formula for $u_n(x)$.

$$\begin{aligned}
u_0(x) &= 6x - 3x^2 \\
u_1(x) &= \int_0^x u_0(t)\,dt = \int_0^x (6t - 3t^2)\,dt = \frac{6x^2}{2} - \frac{3x^3}{3} = 3x^2 - x^3 \\
u_2(x) &= \int_0^x u_1(t)\,dt = \int_0^x \left(\frac{6t^2}{2} - \frac{3t^3}{3}\right) dt = \frac{6x^3}{3\cdot 2} - \frac{3x^4}{4\cdot 3} = x^3 - \frac{x^4}{4} \\
u_3(x) &= \int_0^x u_2(t)\,dt = \int_0^x \left(\frac{6t^3}{3\cdot 2} - \frac{3t^4}{4\cdot 3}\right) dt = \frac{6x^4}{4\cdot 3\cdot 2} - \frac{3x^5}{5\cdot 4\cdot 3} = \frac{x^4}{4} - \frac{x^5}{20} \\
u_4(x) &= \int_0^x u_3(t)\,dt = \int_0^x \left(\frac{6t^4}{4\cdot 3\cdot 2} - \frac{3t^5}{5\cdot 4\cdot 3}\right) dt = \frac{6x^5}{5\cdot 4\cdot 3\cdot 2} - \frac{3x^6}{6\cdot 5\cdot 4\cdot 3} = \frac{x^5}{20} - \frac{x^6}{120} \\
&\vdots \\
u_n(x) &= \int_0^x u_{n-1}(t)\,dt = \int_0^x \left[\frac{6t^n}{n!} - \frac{6t^{n+1}}{(n+1)!}\right] dt = \frac{6x^{n+1}}{(n+1)!} - \frac{6x^{n+2}}{(n+2)!}
\end{aligned}$$

The series for $u(x)$ is telescoping because the second term of every component cancels with the first term of the following component. Only two terms remain in the series: $6x$ and the "last" term. Therefore,

$$u(x) = \sum_{n=0}^{\infty}\left[\frac{6x^{n+1}}{(n+1)!} - \frac{6x^{n+2}}{(n+2)!}\right] = 6x - \underbrace{\lim_{n\to\infty} \frac{6x^{n+2}}{(n+2)!}}_{=0} = 6x.$$