

## Exercise 13

In Exercises 1–26, solve the following Volterra integral equations by using the *Adomian decomposition method*:

$$u(x) = 1 + \int_0^x u(t) dt$$

### Solution

Assume that  $u(x)$  can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= 1 + \int_0^x \sum_{n=0}^{\infty} u_n(t) dt \\ u_0(x) + u_1(x) + u_2(x) + \cdots &= 1 + \int_0^x [u_0(t) + u_1(t) + \cdots] dt \\ u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{1}_{u_0(x)} + \underbrace{\int_0^x u_0(t) dt}_{u_1(x)} + \underbrace{\int_0^x u_1(t) dt}_{u_2(x)} + \cdots \end{aligned}$$

If we set  $u_0(x)$  equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. After enough terms are written, a pattern can be noticed, allowing us to write a general formula for  $u_n(x)$ .

$$\begin{aligned} u_0(x) &= 1 \\ u_1(x) &= \int_0^x u_0(t) dt = \int_0^x (1) dt = \frac{x}{1} \\ u_2(x) &= \int_0^x u_1(t) dt = \int_0^x \left(\frac{t}{1}\right) dt = \frac{x^2}{2 \cdot 1} \\ u_3(x) &= \int_0^x u_2(t) dt = \int_0^x \left(\frac{t^2}{2 \cdot 1}\right) dt = \frac{x^3}{3 \cdot 2 \cdot 1} \\ &\vdots \\ u_n(x) &= \int_0^x u_{n-1}(t) dt = \frac{x^n}{n!} \end{aligned}$$

Therefore,

$$u(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$