Exercise 14

In Exercises 1–26, solve the following Volterra integral equations by using the *Adomian decomposition method*:

$$u(x) = 1 - \int_0^x u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= 1 - \int_0^x \sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= 1 - \int_0^x [u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{1}_{u_0(x)} + \underbrace{\int_0^x (-1)u_0(t)\,dt}_{u_1(x)} + \underbrace{\int_0^x (-1)u_1(t)\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. After enough terms are written, a pattern can be noticed, allowing us to write a general formula for $u_n(x)$.

$$\begin{aligned}
u_0(x) &= 1 \\
u_1(x) &= \int_0^x (-1)u_0(t)\,dt = (-1)\int_0^x (1)\,dt = (-1)\frac{x}{1} \\
u_2(x) &= \int_0^x (-1)u_1(t)\,dt = (-1)^2 \int_0^x \left(\frac{t}{1}\right) dt = (-1)^2 \frac{x^2}{2 \cdot 1} \\
u_3(x) &= \int_0^x (-1)u_2(t)\,dt = (-1)^3 \int_0^x \left(\frac{t^2}{2 \cdot 1}\right) dt = (-1)^3 \frac{x^3}{3 \cdot 2 \cdot 1} \\
&\vdots \\
u_n(x) &= \int_0^x (-1)u_{n-1}(t)\,dt = (-1)^n \frac{x^n}{n!} = \frac{(-x)^n}{n!}
\end{aligned}$$

Therefore,

$$u(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = e^{-x}.$$