

## Exercise 22

In Exercises 1–26, solve the following Volterra integral equations by using the *Adomian decomposition method*:

$$u(x) = 5 + 2x^2 - \int_0^x (x-t)u(t) dt$$

### Solution

Assume that  $u(x)$  can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= 5 + 2x^2 - \int_0^x (x-t) \sum_{n=0}^{\infty} u_n(t) dt \\ u_0(x) + u_1(x) + u_2(x) + \cdots &= 5 + 2x^2 - \int_0^x (x-t)[u_0(t) + u_1(t) + \cdots] dt \\ u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{5 + 2x^2}_{u_0(x)} + \underbrace{\int_0^x (-1)(x-t)u_0(t) dt}_{u_1(x)} + \underbrace{\int_0^x (-1)(x-t)u_1(t) dt}_{u_2(x)} + \cdots \end{aligned}$$

If we set  $u_0(x)$  equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. After enough terms are written, a pattern can be noticed, allowing us to write a general formula for  $u_n(x)$ . Note that the  $(x-t)$  in the integrand essentially means that we integrate the function next to it twice.

$$\begin{aligned} u_0(x) &= 5 + 2x^2 \\ u_1(x) &= \int_0^x (-1)(x-t)u_0(t) dt = (-1) \int_0^x (x-t)(5 + 2t^2) dt = (-1) \left( \frac{5x^2}{2 \cdot 1} + \frac{2x^4}{4 \cdot 3} \right) \\ u_2(x) &= \int_0^x (-1)(x-t)u_1(t) dt = (-1)^2 \int_0^x (x-t) \left( \frac{5t^2}{2 \cdot 1} + \frac{2t^4}{4 \cdot 3} \right) dt \\ &= (-1)^2 \left( \frac{5x^4}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{2x^6}{6 \cdot 5 \cdot 4 \cdot 3} \right) \\ u_3(x) &= \int_0^x (-1)(x-t)u_2(t) dt = (-1)^3 \int_0^x (x-t) \left( \frac{5t^4}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{2t^6}{6 \cdot 5 \cdot 4 \cdot 3} \right) dt \\ &= (-1)^3 \left( \frac{5x^6}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \frac{2x^8}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3} \right) \\ &\vdots \\ u_n(x) &= \int_0^x (x-t)u_{n-1}(t) dt = (-1)^n \left[ \frac{5x^{2n}}{(2n)!} + \frac{4x^{2n+2}}{(2n+2)!} \right] = \frac{5(-1)^n x^{2n}}{(2n)!} + \frac{4(-1)^n x^{2n+2}}{(2n+2)!} \end{aligned}$$

We have

$$\begin{aligned}u(x) &= \sum_{n=0}^{\infty} u_n(x) \\&= \sum_{n=0}^{\infty} \left[ \frac{5(-1)^n x^{2n}}{(2n)!} + \frac{4(-1)^n x^{2n+2}}{(2n+2)!} \right] \\&= 5 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + 4 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!} \\&= 5 \cos x - 4 \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2(n+1)}}{[2(n+1)]!} \\&= 5 \cos x - 4 \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\&= 5 \cos x - 4(\cos x - 1) \\&= 5 \cos x - 4 \cos x + 4.\end{aligned}$$

Therefore,

$$u(x) = \cos x + 4.$$