

Exercise 23

In Exercises 1–26, solve the following Volterra integral equations by using the *Adomian decomposition method*:

$$u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2} \int_0^x (x-t)^2 u(t) dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{2} \int_0^x (x-t)^2 \sum_{n=0}^{\infty} u_n(t) dt \\ u_0(x) + u_1(x) + u_2(x) + \cdots &= 1 + x + \frac{1}{2}x^2 + \frac{1}{2} \int_0^x (x-t)^2 [u_0(t) + u_1(t) + \cdots] dt \\ u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{1 + x + \frac{1}{2}x^2}_{u_0(x)} + \underbrace{\int_0^x \frac{(x-t)^2}{2} u_0(t) dt}_{u_1(x)} + \underbrace{\int_0^x \frac{(x-t)^2}{2} u_1(t) dt + \cdots}_{u_2(x)} \end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. After enough terms are written, a pattern can be noticed, allowing us to write a general formula for $u_n(x)$. Note that the $(x-t)^2/2$ in the integrand essentially means that we integrate the function next to it three times.

$$\begin{aligned} u_0(x) &= 1 + x + \frac{1}{2}x^2 \\ u_1(x) &= \int_0^x \frac{(x-t)^2}{2} u_0(t) dt = \int_0^x \frac{(x-t)^2}{2} \left(1 + t + \frac{1}{2}t^2 \right) dt = \frac{x^3}{3 \cdot 2 \cdot 1} + \frac{x^4}{4 \cdot 3 \cdot 2} + \frac{1}{2} \cdot \frac{x^5}{5 \cdot 4 \cdot 3} \\ u_2(x) &= \int_0^x \frac{(x-t)^2}{2} u_1(t) dt = \int_0^x \frac{(x-t)^2}{2} \left(\frac{t^3}{3 \cdot 2 \cdot 1} + \frac{t^4}{4 \cdot 3 \cdot 2} + \frac{1}{2} \cdot \frac{t^5}{5 \cdot 4 \cdot 3} \right) dt \\ &= \frac{x^6}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \frac{x^7}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} + \frac{1}{2} \cdot \frac{x^8}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3} \\ &\vdots \\ u_n(x) &= \int_0^x (x-t) u_{n-1}(t) dt = \frac{x^{3n}}{(3n)!} + \frac{x^{3n+1}}{(3n+1)!} + \frac{x^{3n+2}}{(3n+2)!} \end{aligned}$$

Therefore,

$$u(x) = \sum_{n=0}^{\infty} \left[\frac{x^{3n}}{(3n)!} + \frac{x^{3n+1}}{(3n+1)!} + \frac{x^{3n+2}}{(3n+2)!} \right] = 1 + x + \frac{x^2}{2!} + \cdots = e^x.$$