Exercise 25

In Exercises 1–26, solve the following Volterra integral equations by using the *Adomian decomposition method*:

$$u(x) = 1 + \frac{1}{2}x + \frac{1}{2}\int_0^x (x - t + 1)u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= 1 + \frac{1}{2}x + \frac{1}{2}\int_0^x (x - t + 1)\sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= 1 + \frac{1}{2}x + \frac{1}{2}\int_0^x (x - t + 1)[u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{1 + \frac{1}{2}x}_{u_0(x)} + \underbrace{\frac{1}{2}\int_0^x (x - t + 1)u_0(t)\,dt}_{u_1(x)} + \underbrace{\frac{1}{2}\int_0^x (x - t + 1)u_1(t)\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. Note that having $(x - t)$ in the integrand essentially means that we integrate the function next to it twice.

$$\begin{aligned}
u_0(x) &= 1 + \frac{1}{2}x \\
u_1(x) &= \frac{1}{2}\int_0^x (x - t + 1)u_0(t)\,dt = \frac{1}{2}\left[\int_0^x (x - t)\left(1 + \frac{1}{2}t\right)dt + \int_0^x \left(1 + \frac{1}{2}t\right)dt\right] = \frac{1}{2}x + \frac{3}{8}x^2 + \frac{1}{24}x^3 \\
u_2(x) &= \frac{1}{2}\int_0^x (x - t + 1)u_1(t)\,dt = \frac{1}{8}x^2 + \frac{5}{48}x^3 + \frac{1}{48}x^4 + \frac{1}{960}x^5 \\
u_3(x) &= \frac{1}{2}\int_0^x (x - t + 1)u_2(t)\,dt = \frac{1}{48}x^3 + \frac{7}{384}x^4 + \frac{3}{640}x^5 + \frac{1}{2304}x^6 + \frac{1}{80640}x^7 \\
&\vdots
\end{aligned}$$

Adding the components together, we find that

$$\begin{aligned}
u(x) &= u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \\
&= 1 + \left(\frac{1}{2} + \frac{1}{2}\right)x + \left(\frac{3}{8} + \frac{1}{8}\right)x^2 + \left(\frac{1}{24} + \frac{5}{48} + \frac{1}{48}\right)x^3 + \cdots \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots,
\end{aligned}$$

which is the beginning of the Taylor series for the exponential function. We venture the guess that $u(x) = e^x$. Substitute this into the integral equation and see if both sides are equal.

$$\begin{aligned}
u(x) &= 1 + \frac{1}{2}x + \frac{1}{2}\int_0^x (x - t + 1)u(t)\,dt \\
e^x &\stackrel{?}{=} 1 + \frac{1}{2}x + \frac{1}{2}\int_0^x (x - t + 1)e^t\,dt \\
&\stackrel{?}{=} 1 + \frac{1}{2}x + \frac{1}{2}\left[\int_0^x (x - t)e^t\,dt + \int_0^x e^t\,dt\right] \\
&\stackrel{?}{=} 1 + \frac{1}{2}x + \frac{1}{2}\left[\int_0^x (e^t - 1)\,dt + e^x - 1\right] \\
&\stackrel{?}{=} 1 + \frac{1}{2}x + \frac{1}{2}(e^x - 1 - x + e^x - 1) \\
&\stackrel{?}{=} \cancel{1} + \cancel{\frac{1}{2}x} + e^x - \cancel{1} - \cancel{\frac{1}{2}x} \\
&= e^x
\end{aligned}$$

Therefore,

$$u(x) = e^x.$$