Exercise 29

In Exercises 27–30, use the *Adomian decomposition method* to find the series solution

$$u(x) = 1 + \frac{1}{2}\int_0^x (x^2 - t^2)u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= 1 + \frac{1}{2}\int_0^x (x^2 - t^2)\sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= 1 + \frac{1}{2}\int_0^x (x^2 - t^2)[u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{1}_{u_0(x)} + \underbrace{\frac{1}{2}\int_0^x (x^2 - t^2)u_0(t)\,dt}_{u_1(x)} + \underbrace{\frac{1}{2}\int_0^x (x^2 - t^2)u_1(t)\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner.

$$\begin{aligned}
u_0(x) &= 1 \\
u_1(x) &= \frac{1}{2}\int_0^x (x^2 - t^2)u_0(t)\,dt = \frac{1}{2}\int_0^x (x^2 - t^2)(1)\,dt = \frac{1}{2}\left(x^3 - \frac{x^3}{3}\right) = \frac{1}{3}x^3 \\
u_2(x) &= \frac{1}{2}\int_0^x (x^2 - t^2)u_1(t)\,dt = \frac{1}{2}\int_0^x (x^2 - t^2)\left(\frac{1}{3}t^3\right)\,dt = \frac{1}{6}\left(\frac{x^6}{4} - \frac{x^6}{6}\right) = \frac{1}{72}x^6 \\
u_3(x) &= \frac{1}{2}\int_0^x (x^2 - t^2)u_2(t)\,dt = \frac{1}{2}\int_0^x (x^2 - t^2)\left(\frac{t^6}{72}\right)\,dt = \frac{1}{144}\left(\frac{x^9}{7} - \frac{x^9}{9}\right) = \frac{1}{4536}x^9 \\
&\ \ \vdots
\end{aligned}$$

Adding the components together, therefore,

$$\begin{aligned}
u(x) &= u_0(x) + u_1(x) + u_2(x) + \cdots \\
&= 1 + \frac{1}{3}x^3 + \frac{1}{72}x^6 + \frac{1}{4536}x^9 + \cdots.
\end{aligned}$$