Exercise 4

In Exercises 1–26, solve the following Volterra integral equations by using the *Adomian decomposition method*:

$$u(x) = x - \frac{2}{3}x^3 - 2\int_0^x u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= x - \frac{2x^3}{3} - 2\int_0^x \sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= x - \frac{2x^3}{3} - 2\int_0^x [u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{x - \frac{2x^3}{3}}_{u_0(x)} + \underbrace{\int_0^x (-2)u_0(t)\,dt}_{u_1(x)} + \underbrace{\int_0^x (-2)u_1(t)\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. After enough terms are written, a pattern can be noticed, allowing us to write a general formula for $u_n(x)$.

$$\begin{aligned}
u_0(x) &= x - \frac{2x^3}{3} \\
u_1(x) &= \int_0^x (-2)u_0(t)\,dt = (-2)\int_0^x \left(t - \frac{2t^3}{3}\right)dt = (-2)\left(\frac{x^2}{2} - \frac{2x^4}{4\cdot 3}\right) \\
u_2(x) &= \int_0^x (-2)u_1(t)\,dt = (-2)^2\int_0^x \left(\frac{t^2}{2} - \frac{2t^4}{4\cdot 3}\right)dt = (-2)^2\left(\frac{x^3}{3\cdot 2} - \frac{2x^5}{5\cdot 4\cdot 3}\right) \\
u_3(x) &= \int_0^x (-2)u_2(t)\,dt = (-2)^3\int_0^x \left(\frac{t^3}{3\cdot 2} - \frac{2t^5}{5\cdot 4\cdot 3}\right)dt = (-2)^3\left(\frac{x^4}{4\cdot 3\cdot 2} - \frac{2x^6}{6\cdot 5\cdot 4\cdot 3}\right) \\
&\vdots \\
u_n(x) &= \int_0^x (-2)u_{n-1}(t)\,dt = (-2)^n\left[\frac{x^{n+1}}{(n+1)!} - \frac{4x^{n+3}}{(n+3)!}\right] = \frac{(-2)^{n+1}x^{n+1}}{(-2)(n+1)!} - \frac{4(-2)^{n+3}x^{n+3}}{(-2)^3(n+3)!}
\end{aligned}$$

Therefore,

$$\begin{aligned}
u(x) &= \sum_{n=0}^{\infty}\left[\frac{(-2x)^{n+1}}{(-2)(n+1)!} - \frac{(-2x)^{n+3}}{(-2)(n+3)!}\right] = \frac{1}{(-2)}\left[\sum_{n=0}^{\infty}\frac{(-2x)^{n+1}}{(n+1)!} - \sum_{n=0}^{\infty}\frac{(-2x)^{n+3}}{(-2)(n+3)!}\right] \\
&= \frac{1}{(-2)}\left[-2x + 2x^2 + \cancel{\sum_{n=0}^{\infty}\frac{(-2x)^{n+3}}{(-2)(n+3)!}} - \cancel{\sum_{n=0}^{\infty}\frac{(-2x)^{n+3}}{(-2)(n+3)!}}\right] = x - x^2.
\end{aligned}$$