

Exercise 5

In Exercises 1–26, solve the following Volterra integral equations by using the *Adomian decomposition method*:

$$u(x) = 1 + x + \int_0^x (x-t)u(t) dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= 1 + x + \int_0^x (x-t) \sum_{n=0}^{\infty} u_n(t) dt \\ u_0(x) + u_1(x) + u_2(x) + \cdots &= 1 + x + \int_0^x (x-t)[u_0(t) + u_1(t) + \cdots] dt \\ u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{1+x}_{u_0(x)} + \underbrace{\int_0^x (x-t)u_0(t) dt}_{u_1(x)} + \underbrace{\int_0^x (x-t)u_1(t) dt}_{u_2(x)} + \cdots \end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. After enough terms are written, a pattern can be noticed, allowing us to write a general formula for $u_n(x)$. Note that the $(x-t)$ in the integrand essentially means that we integrate the function next to it twice.

$$u_0(x) = 1 + x$$

$$u_1(x) = \int_0^x (x-t)u_0(t) dt = \int_0^x (x-t)(1+t) dt = \frac{x^2}{2 \cdot 1} + \frac{x^3}{3 \cdot 2}$$

$$u_2(x) = \int_0^x (x-t)u_1(t) dt = \int_0^x (x-t) \left(\frac{t^2}{2 \cdot 1} + \frac{t^3}{3 \cdot 2} \right) dt = \frac{x^4}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{x^5}{5 \cdot 4 \cdot 3 \cdot 2}$$

$$u_3(x) = \int_0^x (x-t)u_2(t) dt = \int_0^x (x-t) \left(\frac{t^4}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{t^5}{5 \cdot 4 \cdot 3 \cdot 2} \right) dt = \frac{x^6}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \frac{x^7}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}$$

⋮

$$u_n(x) = \int_0^x (x-t)u_{n-1}(t) dt = \frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!}$$

Therefore,

$$\begin{aligned} u(x) &= \sum_{n=0}^{\infty} \left[\frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!} \right] = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \cosh x + \sinh x \\ &= \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x. \end{aligned}$$