

## Exercise 12

Use the *modified decomposition method* to solve the following Volterra integral equations:

$$u(x) = \cos x - (1 - e^{\sin x})x - x \int_0^x e^{\sin t} u(t) dt$$

### Solution

Assume that  $u(x)$  can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= \cos x - (1 - e^{\sin x})x - x \int_0^x e^{\sin t} \sum_{n=0}^{\infty} u_n(t) dt \\ u_0(x) + u_1(x) + u_2(x) + \dots &= \cos x + (-1 + e^{\sin x})x - x \int_0^x e^{\sin t} [u_0(t) + u_1(t) + \dots] dt \\ u_0(x) + u_1(x) + u_2(x) + \dots &= \underbrace{\cos x}_{u_0(x)} + \underbrace{(-1 + e^{\sin x})x - x \int_0^x e^{\sin t} u_0(t) dt}_{u_1(x)} + \underbrace{x \int_0^x e^{\sin t} [-u_1(t)] dt}_{u_2(x)} + \dots \end{aligned}$$

Grouping the terms as we have makes it so that the series terminates early.

$$\begin{aligned} u_0(x) &= \cos x \\ u_1(x) &= (-1 + e^{\sin x})x - x \int_0^x e^{\sin t} u_0(t) dt = (-1 + e^{\sin x})x - x(e^{\sin x} - 1) = 0 \\ u_2(x) &= x \int_0^x e^{\sin t} [-u_1(t)] dt = 0 \\ &\vdots \\ u_n(x) &= x \int_0^x e^{\sin t} [-u_{n-1}(t)] dt = 0, \quad n > 2 \end{aligned}$$

Therefore,

$$u(x) = \cos x.$$