Exercise 6

Use the *modified decomposition method* to solve the following Volterra integral equations:

$$u(x) = e^{-x^2} - \frac{x}{2}(1 - e^{-x^2}) - \int_0^x xtu(t)\,dt$$

[TYPO: In order to get the answer at the back of the book, this minus sign must be a plus sign.]

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= e^{-x^2} + \frac{x}{2}(1 - e^{-x^2}) - \int_0^x xt \sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= e^{-x^2} + \frac{x}{2}(1 - e^{-x^2}) - \int_0^x xt[u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{e^{-x^2}}_{u_0(x)} + \underbrace{\frac{x}{2}(1 - e^{-x^2}) - \int_0^x xtu_0(t)\,dt}_{u_1(x)} + \underbrace{\int_0^x xt[-u_1(t)]\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

Grouping the terms as we have makes it so that the series terminates early.

$$\begin{aligned}
u_0(x) &= e^{-x^2} \\
u_1(x) &= \frac{x}{2}(1 - e^{-x^2}) - \int_0^x xtu_0(t)\,dt = \frac{x}{2}(1 - e^{-x^2}) - \frac{x}{2}(1 - e^{-x^2}) = 0 \\
u_2(x) &= \int_0^x xt[-u_1(t)]\,dt = 0 \\
&\vdots \\
u_n(x) &= \int_0^x xt[-u_{n-1}(t)]\,dt = 0, \quad n > 2
\end{aligned}$$

Therefore,

$$u(x) = e^{-x^2}.$$