Exercise 8

Use the *modified decomposition method* to solve the following Volterra integral equations:

$$u(x) = e^x + xe^x - x - \int_0^x xu(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= e^x + xe^x - x - \int_0^x x \sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= e^x + xe^x - x - \int_0^x x[u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{e^x}_{u_0(x)} + \underbrace{xe^x - x - \int_0^x xu_0(t)\,dt}_{u_1(x)} + \underbrace{\int_0^x x[-u_1(t)]\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

Grouping the terms as we have makes it so that the series terminates early.

$$\begin{aligned}
u_0(x) &= e^x \\
u_1(x) &= xe^x - x - \int_0^x xu_0(t)\,dt = xe^x - x - x(e^x - 1) = 0 \\
u_2(x) &= \int_0^x x[-u_1(t)]\,dt = 0 \\
&\ \vdots \\
u_n(x) &= \int_0^x x[-u_{n-1}(t)]\,dt = 0, \quad n > 2
\end{aligned}$$

Therefore,

$$u(x) = e^x.$$