Exercise 5

Use the *noise terms phenomenon* to solve the following Volterra integral equations:

$$u(x) = -2 + x^2 + \sin x + 2\cos x - \int_0^x (x-t)^2 u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\sum_{n=0}^{\infty} u_n(x) = -2 + x^2 + \sin x + 2\cos x - \int_0^x (x-t)^2 \sum_{n=0}^{\infty} u_n(t)\,dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = -2 + x^2 + \sin x + 2\cos x + (-2)\int_0^x \frac{(x-t)^2}{2}[u_0(t) + u_1(t) + \cdots]\,dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \underbrace{-2 + x^2 + \sin x + 2\cos x}_{u_0(x)} + \underbrace{(-2)\int_0^x \frac{(x-t)^2}{2} u_0(t)\,dt}_{u_1(x)} + \underbrace{(-2)\int_0^x \frac{(x-t)^2}{2} u_1(t)\,dt}_{u_2(x)} + \cdots$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. Note that having $(x-t)^2/2$ in the integrand means we integrate the function next to it three times (from 0 to x).

$$\begin{aligned}
u_0(x) &= -2 + x^2 + \sin x + 2\cos x \\
u_1(x) &= -2\int_0^x \frac{(x-t)^2}{2} u_0(t)\,dt = -2\left[-\frac{x^3}{3} + \frac{x^5}{60} + \left(\cos x + \frac{x^2}{2} - 1\right) + 2(x - \sin x)\right] \\
&= 2 - 4x - x^2 + \frac{2x^3}{3} - \frac{x^5}{30} - 2\cos x + 4\sin x \\
&\vdots
\end{aligned}$$

The noise terms, ∓ 2 and $\pm x^2$ and $\pm 2\cos x$, appear in both $u_0(x)$ and $u_1(x)$. Cancelling -2, x^2, and $2\cos x$ from $u_0(x)$ leaves $\sin x$. Now we check to see whether $u(x) = \sin x$ satisfies the integral equation.

$$\sin x \stackrel{?}{=} -2 + x^2 + \sin x + 2\cos x - \int_0^x (x-t)^2 \sin t\,dt$$

$$\sin x \stackrel{?}{=} \cancel{-2} + \cancel{x^2} + \sin x + \cancel{2\cos x} - 2\left(\cancel{\cos x} + \cancel{\frac{x^2}{2}} - \cancel{1}\right)$$

$$\sin x = \sin x$$

Therefore,

$$u(x) = \sin x.$$