Exercise 8

Use the *noise terms phenomenon* to solve the following Volterra integral equations:

$$u(x) = x + \cosh x + x^2 \sinh x - x\cosh x - \int_0^x xtu(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\sum_{n=0}^{\infty} u_n(x) = x + \cosh x + x^2 \sinh x - x\cosh x - \int_0^x xt \sum_{n=0}^{\infty} u_n(t)\,dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = x + \cosh x + x^2 \sinh x - x\cosh x - \int_0^x xt[u_0(t) + u_1(t) + \cdots]\,dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \underbrace{x + \cosh x + x^2 \sinh x - x\cosh x}_{u_0(x)} + \underbrace{x\int_0^x [-tu_0(t)]\,dt}_{u_1(x)} + \underbrace{x\int_0^x [-tu_1(t)]\,dt}_{u_2(x)} + \cdots$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner.

$$u_0(x) = x + \cosh x + x^2 \sinh x - x\cosh x$$

$$u_1(x) = x\int_0^x [-tu_0(t)]\,dt = -x\int_0^x t(t + \cosh t + t^2 \sinh t - t\cosh t)\,dt$$

$$= -x - \frac{x^4}{3} + x\cosh x - 8x^2\cosh x - x^4\cosh x + 8x\sinh x - x^2\sinh x + 4x^3\sinh x$$

$$\vdots$$

The noise terms, $\pm x$ and $\pm x^2 \sinh x$ and $\mp x\cosh x$, appear in both $u_0(x)$ and $u_1(x)$. Cancelling x, $x^2\sinh x$, and $-x\cosh x$ from $u_0(x)$ leaves $\cosh x$. Now we check to see whether $u(x) = \cosh x$ satisfies the integral equation.

$$\cosh x \stackrel{?}{=} x + \cosh x + x^2\sinh x - x\cosh x - \int_0^x xt\cosh t\,dt$$

$$\cosh x \stackrel{?}{=} \cancel{x} + \cosh x + \cancel{x^2\sinh x} - \cancel{x\cosh x} - x(\cancel{1} - \cancel{\cosh x} + \cancel{x\sinh x})$$

$$\cosh x = \cosh x$$

Therefore,

$$u(x) = \cosh x.$$