Exercise 3

Use the *Laplace transform method* to solve the Volterra integral equations:

$$u(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u(t)\,dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)\,dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L}\left\{\int_0^x f(x-t)g(t)\,dt\right\}$$

Take the Laplace transform of both sides of the integral equation.

$$\begin{aligned}
\mathcal{L}\{u(x)\} &= \mathcal{L}\left\{1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u(t)\,dt\right\} \\
U(s) &= \mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{x^2\} + \frac{1}{6}\mathcal{L}\left\{\int_0^x (x-t)^3 u(t)\,dt\right\} \\
&= \mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{x^2\} + \frac{1}{6}\mathcal{L}\{x^3\}U(s) \\
&= \frac{1}{s} - \frac{1}{2}\left(\frac{2}{s^3}\right) + \frac{1}{6}\left(\frac{6}{s^4}\right)U(s) \\
&= \frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^4}U(s)
\end{aligned}$$

Solve for $U(s)$.

$$\begin{aligned}
\left(1 - \frac{1}{s^4}\right)U(s) &= \frac{1}{s} - \frac{1}{s^3} \\
U(s) &= \frac{\frac{1}{s} - \frac{1}{s^3}}{1 - \frac{1}{s^4}} \\
&= \frac{s^3 - s}{s^4 - 1} \\
&= \frac{s(s^2-1)}{(s^2+1)(s^2-1)} \\
&= \frac{s}{s^2+1}
\end{aligned}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$\begin{aligned}
u(x) &= \mathcal{L}^{-1}\{U(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \\
&= \cos x
\end{aligned}$$