# Exercise 8

Use the *Laplace transform method* to solve the Volterra integral equations:

$$u(x) = 1 - \int_0^x \left((x-t)^2 - 1\right) u(t)\, dt$$

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### Solution

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)\, dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L}\left\{\int_0^x f(x-t)g(t)\, dt\right\}$$

Take the Laplace transform of both sides of the integral equation.

$$\begin{aligned}
\mathcal{L}\{u(x)\} &= \mathcal{L}\left\{1 - \int_0^x \left((x-t)^2 - 1\right) u(t)\, dt\right\} \\
U(s) &= \mathcal{L}\{1\} - \mathcal{L}\left\{\int_0^x \left((x-t)^2 - 1\right) u(t)\, dt\right\} \\
&= \mathcal{L}\{1\} - \mathcal{L}\{x^2 - 1\}U(s) \\
&= \mathcal{L}\{1\} - (\mathcal{L}\{x^2\} - \mathcal{L}\{1\})U(s) \\
&= \frac{1}{s} - \left(\frac{2}{s^3} - \frac{1}{s}\right) U(s)
\end{aligned}$$

Solve for $U(s)$.

$$\begin{aligned}
\left(1 + \frac{2}{s^3} - \frac{1}{s}\right) U(s) &= \frac{1}{s} \\
(s^3 + 2 - s^2)U(s) &= s^2 \\
U(s) &= \frac{s^2}{s^3 - s^2 + 2} \\
&= \frac{s^2}{(s+1)(s^2 - 2s + 2)} \\
&= \frac{\frac{1}{5}}{s+1} + \frac{\frac{4}{5}s - \frac{2}{5}}{s^2 - 2s + 2} \\
&= \frac{1}{5}\left(\frac{1}{s+1}\right) + \frac{4}{5}\left[\frac{s - \frac{1}{2}}{(s-1)^2 + 1}\right] \\
&= \frac{1}{5}\left(\frac{1}{s+1}\right) + \frac{4}{5}\left[\frac{s-1}{(s-1)^2 + 1} + \frac{\frac{1}{2}}{(s-1)^2 + 1}\right]
\end{aligned}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$\begin{aligned}
u(x) &= \mathcal{L}^{-1}\{U(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{5}\left(\frac{1}{s+1}\right) + \frac{4}{5}\left[\frac{s-1}{(s-1)^2+1} + \frac{\frac{1}{2}}{(s-1)^2+1}\right]\right\} \\
&= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{4}{5}\left[\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}\right] \\
&= \frac{1}{5}e^{-x} + \frac{4}{5}\left(e^x\cos x + \frac{1}{2}e^x\sin x\right)
\end{aligned}$$

Therefore,

$$u(x) = \frac{1}{5}\left[e^{-x} + 2e^x(2\cos x + \sin x)\right].$$