

Exercise 10

Use the *series solution method* to solve the Volterra integral equations:

$$u(x) = x \cosh x - \int_0^x tu(t) dt$$

Solution

We seek a series solution for u :

$$u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

Substitute this and the Taylor series expansion of $\cosh x$,

$$\cosh x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \dots,$$

into the integral equation.

$$\begin{aligned} & a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \dots \\ &= x \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \dots \right) \\ & \quad - \int_0^x t(a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + \dots) dt \end{aligned}$$

$$\begin{aligned} & a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \dots \\ &= x + \frac{1}{2}x^3 + \frac{1}{24}x^5 + \frac{1}{720}x^7 + \dots \\ & \quad - \frac{a_0}{2}x^2 - \frac{a_1}{3}x^3 - \frac{a_2}{4}x^4 - \frac{a_3}{5}x^5 - \frac{a_4}{6}x^6 - \frac{a_5}{7}x^7 - \dots \end{aligned}$$

$$\begin{aligned} & a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \dots \\ &= x - \frac{a_0}{2}x^2 + \left(\frac{1}{2} - \frac{a_1}{3} \right) x^3 - \frac{a_2}{4}x^4 \\ & \quad + \left(\frac{1}{24} - \frac{a_3}{5} \right) x^5 - \frac{a_4}{6}x^6 + \left(\frac{1}{720} - \frac{a_5}{7} \right) x^7 + \dots \end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i .

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_2 &= -\frac{a_0}{2} && \rightarrow && a_2 = 0 \\ a_3 &= \frac{1}{2} - \frac{a_1}{3} && \rightarrow && a_3 = \frac{1}{6} \\ a_4 &= -\frac{a_2}{4} && \rightarrow && a_4 = 0 \\ a_5 &= \frac{1}{24} - \frac{a_3}{5} && \rightarrow && a_5 = \frac{1}{120} \\ a_6 &= -\frac{a_4}{6} && \rightarrow && a_6 = 0 \\ a_7 &= \frac{1}{720} - \frac{a_5}{7} && \rightarrow && a_7 = \frac{1}{5040} \\ &\vdots && && \vdots \end{aligned}$$

So then

$$\begin{aligned} u(x) &= x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \cdots \\ &= \sinh x. \end{aligned}$$