Exercise 13

Use the *series solution method* to solve the Volterra integral equations:

$$u(x) = x - x\ln(1+x) + \int_0^x u(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots.$$

Substitute this and the Taylor series expansion of $\ln(1+x)$,

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots,$$

into the integral equation.

$$\begin{aligned}
a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\
= x - x\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots\right) \\
+ \int_0^x (a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 + \cdots)\,dt
\end{aligned}$$

$$\begin{aligned}
a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\
= x - x^2 + \frac{1}{2}x^3 - \frac{1}{3}x^4 + \frac{1}{4}x^5 - \frac{1}{5}x^6 + \frac{1}{6}x^7 - \cdots \\
+ a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \frac{a_4}{5}x^5 + \frac{a_5}{6}x^6 + \frac{a_6}{7}x^7 + \cdots
\end{aligned}$$

$$\begin{aligned}
a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\
= (1+a_0)x + \left(-1 + \frac{a_1}{2}\right)x^2 + \left(\frac{1}{2} + \frac{a_2}{3}\right)x^3 + \left(-\frac{1}{3} + \frac{a_3}{4}\right)x^4 \\
+ \left(\frac{1}{4} + \frac{a_4}{5}\right)x^5 + \left(-\frac{1}{5} + \frac{a_5}{6}\right)x^6 + \left(\frac{1}{6} + \frac{a_6}{7}\right)x^7 + \cdots
\end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$
\begin{aligned}
a_0 &= 0 \\
a_1 &= 1 + a_0 & &\rightarrow & a_1 &= 1 \\
a_2 &= -1 + \frac{a_1}{2} & &\rightarrow & a_2 &= -\frac{1}{2} \\
a_3 &= \frac{1}{2} + \frac{a_2}{3} & &\rightarrow & a_3 &= \frac{1}{3} \\
a_4 &= -\frac{1}{3} + \frac{a_3}{4} & &\rightarrow & a_4 &= -\frac{1}{4} \\
a_5 &= \frac{1}{4} + \frac{a_4}{5} & &\rightarrow & a_5 &= \frac{1}{5} \\
a_6 &= -\frac{1}{5} + \frac{a_5}{6} & &\rightarrow & a_6 &= -\frac{1}{6} \\
a_7 &= \frac{1}{6} + \frac{a_6}{7} & &\rightarrow & a_7 &= \frac{1}{7} \\
&\vdots & & & &\vdots
\end{aligned}
$$

So then

$$
\begin{aligned}
u(x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \frac{1}{7}x^7 - \cdots \\
&= \ln(1 + x).
\end{aligned}
$$