# Exercise 14

Use the *series solution method* to solve the Volterra integral equations:

$$u(x) = x^2 - \frac{1}{2}x^3 + x^3 \ln(1+x) - \int_0^x 2xu(t)\,dt$$

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### Solution

We seek a series solution for $u$:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Substitute this and the Taylor series expansion of $\ln(1+x)$,

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots ,$$

into the integral equation.

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9 + \cdots \\ = x^2 - \frac{1}{2}x^3 + x^3\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots\right) \\ - \int_0^x 2x(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + \cdots)\,dt \end{aligned}$$

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9 + \cdots \\ = x^2 - \frac{1}{2}x^3 + x^4 - \frac{1}{2}x^5 + \frac{1}{3}x^6 - \frac{1}{4}x^7 + \frac{1}{5}x^8 - \frac{1}{6}x^9 + \cdots \\ - 2x\left(a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \frac{a_4}{5}x^5 + \frac{a_5}{6}x^6 + \frac{a_6}{7}x^7 + \frac{a_7}{8}x^8 + \cdots\right) \end{aligned}$$

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9 + \cdots \\ = x^2 - \frac{1}{2}x^3 + x^4 - \frac{1}{2}x^5 + \frac{1}{3}x^6 - \frac{1}{4}x^7 + \frac{1}{5}x^8 - \frac{1}{6}x^9 + \cdots \\ - 2a_0 x^2 - a_1 x^3 - \frac{2a_2}{3}x^4 - \frac{a_3}{2}x^5 - \frac{2a_4}{5}x^6 - \frac{a_5}{3}x^7 - \frac{2a_6}{7}x^8 - \frac{a_7}{4}x^9 - \cdots \end{aligned}$$

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9 + \cdots \\ = (1 - 2a_0)x^2 + \left(-\frac{1}{2} - a_1\right)x^3 + \left(1 - \frac{2a_2}{3}\right)x^4 + \left(-\frac{1}{2} - \frac{a_3}{2}\right)x^5 \\ + \left(\frac{1}{3} - \frac{2a_4}{5}\right)x^6 + \left(-\frac{1}{4} - \frac{a_5}{3}\right)x^7 + \left(\frac{1}{5} - \frac{2a_6}{7}\right)x^8 + \left(-\frac{1}{6} - \frac{a_7}{4}\right)x^9 + \cdots \end{aligned}$$

Match the coefficients of the respective powers of $x$ to determine $a_i$.

$$
\begin{aligned}
a_0 &= 0 \\
a_1 &= 0 \\
a_2 &= 1 - 2a_0 & \rightarrow \quad & a_2 = 1 \\
a_3 &= -\frac{1}{2} - a_1 & \rightarrow \quad & a_3 = -\frac{1}{2} \\
a_4 &= 1 - \frac{2a_2}{3} & \rightarrow \quad & a_4 = \frac{1}{3} \\
a_5 &= -\frac{1}{2} - \frac{a_3}{2} & \rightarrow \quad & a_5 = -\frac{1}{4} \\
a_6 &= \frac{1}{3} - \frac{2a_4}{5} & \rightarrow \quad & a_6 = \frac{1}{5} \\
a_7 &= -\frac{1}{4} - \frac{a_5}{3} & \rightarrow \quad & a_7 = -\frac{1}{6} \\
a_8 &= \frac{1}{5} - \frac{2a_6}{7} & \rightarrow \quad & a_8 = \frac{1}{7} \\
a_9 &= -\frac{1}{6} - \frac{a_7}{4} & \rightarrow \quad & a_9 = -\frac{1}{8} \\
&\vdots & & \vdots
\end{aligned}
$$

So then

$$
\begin{aligned}
u(x) &= x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \frac{1}{4}x^5 + \frac{1}{5}x^6 - \frac{1}{6}x^7 + \frac{1}{7}x^8 - \frac{1}{8}x^9 + \cdots \\
&= x\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \frac{1}{7}x^7 - \frac{1}{8}x^8 + \cdots\right) \\
&= x\ln(1+x).
\end{aligned}
$$