Exercise 3

Use the *series solution method* to solve the Volterra integral equations:

$$u(x) = x + \int_0^x (x-t)u(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Substitute this into the integral equation.

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \cdots &= x + \int_0^x (x-t)(a_0 + a_1 t + a_2 t^2 + \cdots)\,dt \\ &= x + \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 \\ &\quad + \frac{a_3}{20}x^5 + \frac{a_4}{30}x^6 + \frac{a_5}{42}x^7 + \cdots \end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_2 = \frac{a_0}{2} &\quad \rightarrow \quad a_2 = 0 \\ a_3 = \frac{a_1}{6} &\quad \rightarrow \quad a_3 = \frac{1}{6} \\ a_4 = \frac{a_2}{12} &\quad \rightarrow \quad a_4 = 0 \\ a_5 = \frac{a_3}{20} &\quad \rightarrow \quad a_5 = \frac{1}{120} \\ a_6 = \frac{a_4}{30} &\quad \rightarrow \quad a_6 = 0 \\ a_7 = \frac{a_5}{42} &\quad \rightarrow \quad a_7 = \frac{1}{5040} \\ \vdots &\qquad\qquad\quad \vdots \end{aligned}$$

So then

$$\begin{aligned} u(x) &= x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \cdots \\ &= \sinh x. \end{aligned}$$