Exercise 4

Use the *series solution method* to solve the Volterra integral equations:

$$u(x) = 1 + \frac{1}{2}x + \frac{1}{2}\int_0^x (x - t + 1)u(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots.$$

Substitute this into the integral equation.

$$\begin{aligned} a_0 &+ a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\ &= 1 + \frac{1}{2}x + \frac{1}{2}\int_0^x (x - t + 1)(a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + \cdots)\,dt \end{aligned}$$

$$\begin{aligned} a_0 &+ a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\ &= 1 + \frac{1}{2}x + \frac{1}{2}\int_0^x (x - t)(a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + \cdots)\,dt \\ &\quad + \frac{1}{2}\int_0^x (a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 + \cdots)\,dt \end{aligned}$$

$$\begin{aligned} a_0 &+ a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\ &= 1 + \frac{1}{2}x + \frac{1}{2}\left(\frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 + \frac{a_3}{20}x^5 + \frac{a_4}{30}x^6 + \frac{a_5}{42}x^7 + \cdots\right) \\ &\quad + \frac{1}{2}\left(a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \frac{a_4}{5}x^5 + \frac{a_5}{6}x^6 + \frac{a_6}{7}x^7 + \cdots\right) \end{aligned}$$

$$\begin{aligned} a_0 &+ a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\ &= 1 + \frac{1}{2}(1 + a_0)x + \frac{1}{4}(a_0 + a_1)x^2 + \frac{1}{12}(a_1 + 2a_2)x^3 + \frac{1}{24}(a_2 + 3a_3)x^4 \\ &\quad + \frac{1}{40}(a_3 + 4a_4)x^5 + \frac{1}{60}(a_4 + 5a_5)x^6 + \frac{1}{84}(a_5 + 6a_6)x^7 + \cdots \end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= \frac{1}{2}(1 + a_0) && \rightarrow && a_1 = 1 \\
a_2 &= \frac{1}{4}(a_0 + a_1) && \rightarrow && a_2 = \frac{1}{2} \\
a_3 &= \frac{1}{12}(a_1 + 2a_2) && \rightarrow && a_3 = \frac{1}{6} \\
a_4 &= \frac{1}{24}(a_2 + 3a_3) && \rightarrow && a_4 = \frac{1}{24} \\
a_5 &= \frac{1}{40}(a_3 + 4a_4) && \rightarrow && a_5 = \frac{1}{120} \\
a_6 &= \frac{1}{60}(a_4 + 5a_5) && \rightarrow && a_6 = \frac{1}{720} \\
a_7 &= \frac{1}{84}(a_5 + 6a_6) && \rightarrow && a_7 = \frac{1}{5040} \\
&\vdots && && \vdots
\end{aligned}$$

So then

$$\begin{aligned}
u(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \cdots \\
&= e^x.
\end{aligned}$$