Exercise 8

Use the *series solution method* to solve the Volterra integral equations:

$$u(x) = 1 + 2\sin x - \int_0^x u(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots.$$

Substitute this and the Taylor series expansion of $\sin x$,

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots,$$

into the integral equation.

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\ = 1 + 2\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots\right) \\ - \int_0^x (a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 + \cdots)\,dt \end{aligned}$$

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\ = 1 + 2x - \frac{1}{3}x^3 + \frac{1}{60}x^5 - \frac{1}{2520}x^7 + \cdots \\ - a_0x - \frac{a_1}{2}x^2 - \frac{a_2}{3}x^3 - \frac{a_3}{4}x^4 - \frac{a_4}{5}x^5 - \frac{a_5}{6}x^6 - \frac{a_6}{7}x^7 - \cdots \end{aligned}$$

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\ = 1 + (2 - a_0)x - \frac{a_1}{2}x^2 - \frac{1}{3}(1 + a_2)x^3 - \frac{a_3}{4}x^4 \\ + \frac{1}{60}(1 - 12a_4)x^5 - \frac{a_5}{6}x^6 - \frac{1}{2520}(1 + 360a_6)x^7 + \cdots \end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= 2 - a_0 & \rightarrow \quad & a_1 = 1 \\
a_2 &= -\frac{a_1}{2} & \rightarrow \quad & a_2 = -\frac{1}{2} \\
a_3 &= -\frac{1}{3}(1 + a_2) & \rightarrow \quad & a_3 = -\frac{1}{6} \\
a_4 &= -\frac{a_3}{4} & \rightarrow \quad & a_4 = \frac{1}{24} \\
a_5 &= \frac{1}{60}(1 - 12a_4) & \rightarrow \quad & a_5 = \frac{1}{120} \\
a_6 &= -\frac{a_5}{6} & \rightarrow \quad & a_6 = -\frac{1}{720} \\
a_7 &= -\frac{1}{2520}(1 + 360a_6) & \rightarrow \quad & a_7 = -\frac{1}{5040} \\
&\vdots & & \vdots
\end{aligned}
$$

So then

$$
\begin{aligned}
u(x) &= 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 - \frac{1}{720}x^6 - \frac{1}{5040}x^7 + \cdots \\
&= \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots\right) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots\right) \\
&= \cos x + \sin x.
\end{aligned}
$$