Exercise 10

Use the *series solution method* to solve the Volterra integral equations of the first kind:

$$1 - x - e^{-x} = \int_0^x (x-t)u(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

Substitute this and the Taylor series expansion of e^{-x},

$$e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \cdots,$$

into the integral equation.

$$\begin{aligned} 1 - x - \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \frac{1}{720}x^6 - \frac{1}{5040}x^7 + \cdots\right) \\ = \int_0^x (x-t)(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \cdots)\,dt \end{aligned}$$

Simplify the left side and evaluate the integral on the right side.

$$\begin{aligned} -\frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{1}{120}x^5 - \frac{1}{720}x^6 + \frac{1}{5040}x^7 - \cdots \\ = \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 + \frac{a_3}{20}x^5 + \frac{a_4}{30}x^6 + \frac{a_5}{42}x^7 + \cdots \end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$\begin{array}{lll} \dfrac{a_0}{2} = -\dfrac{1}{2} & \dfrac{a_3}{20} = \dfrac{1}{120} & \cdots \\ \dfrac{a_1}{6} = \dfrac{1}{6} & \dfrac{a_4}{30} = -\dfrac{1}{720} & \cdots \\ \dfrac{a_2}{12} = -\dfrac{1}{24} & \dfrac{a_5}{42} = \dfrac{1}{5040} & \cdots \end{array}$$

So then

$$\begin{array}{lll} a_0 = -1 & a_3 = \dfrac{1}{6} & \cdots \\ a_1 = 1 & a_4 = -\dfrac{1}{24} & \cdots \\ a_2 = -\dfrac{1}{2} & a_5 = \dfrac{1}{120} & \cdots \end{array}$$

and

$$\begin{aligned} u(x) &= -1 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{1}{120}x^5 - \cdots \\ &= -\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \cdots\right) \\ &= -e^{-x}. \end{aligned}$$