Exercise 12

Use the *series solution method* to solve the Volterra integral equations of the first kind:

$$\frac{1}{2}x^2e^x = \int_0^x e^{x-t}u(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots.$$

Substitute this and the Taylor series expansion of e^x,

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots,$$

into the integral equation.

$$\begin{aligned}
&\frac{1}{2}x^2\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right) \\
&\qquad = \int_0^x \left[1 + (x-t) + \frac{1}{2}(x-t)^2 + \frac{1}{6}(x-t)^3 + \cdots\right](a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + \cdots)\,dt
\end{aligned}$$

Simplify the left side and evaluate the integral on the right side.

$$\begin{aligned}
\frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{1}{12}x^5 + \cdots &= \int_0^x (a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + \cdots)\,dt \\
&\quad + \int_0^x (x-t)(a_0 + a_1t + a_2t^2 + a_3t^3 + \cdots)\,dt \\
&\quad + \int_0^x \frac{1}{2}(x-t)^2(a_0 + a_1t + a_2t^2 + \cdots)\,dt \\
&\quad + \int_0^x \frac{1}{6}(x-t)^3(a_0 + a_1t + \cdots)\,dt \\
&\quad + \int_0^x \frac{1}{24}(x-t)^4(a_0 + \cdots)\,dt \\
&\quad + \cdots \\
&= a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \frac{a_4}{5}x^5 + \cdots \\
&\quad + \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 + \frac{a_3}{20}x^5 + \cdots \\
&\quad + \frac{a_0}{6}x^3 + \frac{a_1}{24}x^4 + \frac{a_2}{60}x^5 + \cdots \\
&\quad + \frac{a_0}{24}x^4 + \frac{a_1}{120}x^5 + \cdots \\
&\quad + \frac{a_0}{120}x^5 + \cdots \\
&\quad + \cdots
\end{aligned}$$

Combine like-terms on the right side.

$$\begin{aligned}\frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{1}{12}x^5 + \cdots \\ = a_0 x + \frac{1}{2}(a_0 + a_1)x^2 + \frac{1}{6}(a_0 + a_1 + 2a_2)x^3 + \frac{1}{24}(a_0 + a_1 + 2a_2 + 6a_3)x^4 \\ + \frac{1}{120}(a_0 + a_1 + 2a_2 + 6a_3 + 24a_4)x^5 + \cdots\end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$\begin{aligned} a_0 &= 0 \\ \frac{1}{2}(a_0 + a_1) &= \frac{1}{2} \quad \rightarrow \quad a_1 = 1 \\ \frac{1}{6}(a_0 + a_1 + 2a_2) &= \frac{1}{2} \quad \rightarrow \quad a_2 = 1 \\ \frac{1}{24}(a_0 + a_1 + 2a_2 + 6a_3) &= \frac{1}{4} \quad \rightarrow \quad a_3 = \frac{1}{2} \\ \frac{1}{120}(a_0 + a_1 + 2a_2 + 6a_3 + 24a_4) &= \frac{1}{12} \quad \rightarrow \quad a_4 = \frac{1}{6} \\ &\vdots \end{aligned}$$

So then

$$\begin{aligned} u(x) &= x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \cdots \\ &= x\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right) \\ &= xe^x. \end{aligned}$$