Exercise 6

Use the *series solution method* to solve the Volterra integral equations of the first kind:

$$-x + 2\sin x - x\cos x = \int_0^x (x-t)u(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Substitute this and the Taylor series expansions of $\cos x$,

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40\,320}x^8 + \cdots ,$$

and $\sin x$,

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362\,880}x^9 + \cdots ,$$

into the integral equation.

$$\begin{aligned} -x + 2\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362\,880}x^9 + \cdots\right) \\ - x\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40\,320}x^8 + \cdots\right) \\ = \int_0^x (x-t)(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots)\,dt \end{aligned}$$

Simplify the left side and evaluate the integral on the right side.

$$\begin{aligned} (-1+2-1)x + \left(-\frac{1}{3} + \frac{1}{2}\right)x^3 + \left(\frac{1}{60} - \frac{1}{24}\right)x^5 \\ + \left(-\frac{1}{2520} + \frac{1}{720}\right)x^7 + \left(\frac{1}{181\,440} - \frac{1}{40\,320}\right)x^9 + \cdots \\ = \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 + \frac{a_3}{20}x^5 + \cdots \end{aligned}$$

$$\begin{aligned} \frac{1}{6}x^3 - \frac{1}{40}x^5 + \frac{1}{1008}x^7 - \frac{1}{51\,840}x^9 \\ = \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 + \frac{a_3}{20}x^5 + \frac{a_4}{30}x^6 + \frac{a_5}{42}x^7 + \frac{a_6}{56}x^8 + \frac{a_7}{72}x^9 + \cdots \end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$\begin{array}{llll} \frac{a_0}{2} = 0 & \frac{a_3}{20} = -\frac{1}{40} & \frac{a_6}{56} = 0 & \cdots \\ \frac{a_1}{6} = \frac{1}{6} & \frac{a_4}{30} = 0 & \frac{a_7}{72} = -\frac{1}{51\,840} & \cdots \\ \frac{a_2}{12} = 0 & \frac{a_5}{42} = \frac{1}{1008} & & \cdots \end{array}$$

So then

$$\begin{array}{llll} a_0 = 0 & a_3 = -\frac{1}{2} & a_6 = 0 & \cdots \\ a_1 = 1 & a_4 = 0 & a_7 = -\frac{1}{720} & \cdots \\ a_2 = 0 & a_5 = \frac{1}{24} & & \cdots \end{array}$$

and

$$\begin{aligned} u(x) &= x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \frac{1}{720}x^7 + \cdots \\ &= x\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots\right) \\ &= x\cos x. \end{aligned}$$