Exercise 7

Use the *series solution method* to solve the Volterra integral equations of the first kind:

$$-1 + \cosh x = \int_0^x (x - t)u(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Substitute this and the Taylor series expansion of $\cosh x$,

$$\cosh x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \frac{1}{40\,320}x^8 + \cdots ,$$

into the integral equation.

$$\begin{aligned} -1 + \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \frac{1}{40\,320}x^8 + \cdots\right) \\ = \int_0^x (x - t)(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \cdots)\,dt \end{aligned}$$

Simplify the left side and evaluate the integral on the right side.

$$\begin{aligned} \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \frac{1}{40\,320}x^8 + \cdots \\ = \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 + \frac{a_3}{20}x^5 + \frac{a_4}{30}x^6 + \frac{a_5}{42}x^7 + \frac{a_6}{56}x^8 + \cdots \end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$\begin{array}{llll} \frac{a_0}{2} = \frac{1}{2} & \frac{a_3}{20} = 0 & \frac{a_6}{56} = \frac{1}{40\,320} & \cdots \\ \frac{a_1}{6} = 0 & \frac{a_4}{30} = \frac{1}{720} & & \cdots \\ \frac{a_2}{12} = \frac{1}{24} & \frac{a_5}{42} = 0 & & \cdots \end{array}$$

So then

$$\begin{array}{llll} a_0 = 1 & a_3 = 0 & a_6 = \frac{1}{720} & \cdots \\ a_1 = 0 & a_4 = \frac{1}{24} & & \cdots \\ a_2 = \frac{1}{2} & a_5 = 0 & & \cdots \end{array}$$

and

$$\begin{aligned} u(x) &= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \cdots \\ &= \cosh x. \end{aligned}$$