

Exercise 8

Use the *series solution method* to solve the Volterra integral equations of the first kind:

$$x - 2 \sinh x + x \cosh x = \int_0^x (x-t)u(t) dt$$

Solution

We seek a series solution for u :

$$u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

Substitute this and the Taylor series expansions of $\cosh x$,

$$\cosh x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \frac{1}{40320}x^8 + \dots,$$

and $\sinh x$,

$$\sinh x = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \frac{1}{362880}x^9 + \dots,$$

into the integral equation.

$$\begin{aligned} x - 2 \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \frac{1}{362880}x^9 + \dots \right) \\ + x \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \frac{1}{40320}x^8 + \dots \right) \\ = \int_0^x (x-t)(a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 + a_7t^7 + \dots) dt \end{aligned}$$

Simplify the left side and evaluate the integral on the right side.

$$\begin{aligned} (1 - 2 + 1)x + \left(-\frac{1}{3} + \frac{1}{2} \right) x^3 + \left(-\frac{1}{60} + \frac{1}{24} \right) x^5 \\ + \left(-\frac{1}{2520} + \frac{1}{720} \right) x^7 + \left(-\frac{1}{181440} + \frac{1}{40320} \right) x^9 + \dots \\ = \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 + \frac{a_3}{20}x^5 + \frac{a_4}{30}x^6 + \frac{a_5}{42}x^7 + \frac{a_6}{56}x^8 + \frac{a_7}{72}x^9 + \dots \end{aligned}$$

$$\begin{aligned} \frac{1}{6}x^3 + \frac{1}{40}x^5 + \frac{1}{1008}x^7 + \frac{1}{51840}x^9 + \dots \\ = \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 + \frac{a_3}{20}x^5 + \frac{a_4}{30}x^6 + \frac{a_5}{42}x^7 + \frac{a_6}{56}x^8 + \frac{a_7}{72}x^9 + \dots \end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i .

$$\begin{array}{lll} \frac{a_0}{2} = 0 & \frac{a_3}{20} = \frac{1}{40} & \frac{a_6}{56} = 0 \quad \dots \\ \frac{a_1}{6} = \frac{1}{6} & \frac{a_4}{30} = 0 & \frac{a_7}{72} = \frac{1}{51840} \quad \dots \\ \frac{a_2}{12} = 0 & \frac{a_5}{42} = \frac{1}{1008} & \dots \end{array}$$

So then

$$\begin{array}{llll} a_0 = 0 & a_3 = \frac{1}{2} & a_6 = 0 & \dots \\ a_1 = 1 & a_4 = 0 & a_7 = \frac{1}{720} & \dots \\ a_2 = 0 & a_5 = \frac{1}{24} & & \dots \end{array}$$

and

$$\begin{aligned} u(x) &= x + \frac{1}{2}x^3 + \frac{1}{24}x^5 + \frac{1}{720}x^7 + \dots \\ &= x \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \dots \right) \\ &= x \cosh x. \end{aligned}$$