

## Exercise 9

Use the *series solution method* to solve the Volterra integral equations of the first kind:

$$1 + x - \sin x - \cos x = \int_0^x (x-t)u(t) dt$$

### Solution

We seek a series solution for  $u$ :

$$u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

Substitute this and the Taylor series expansions of  $\cos x$ ,

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 + \dots,$$

and  $\sin x$ ,

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 + \dots,$$

into the integral equation.

$$\begin{aligned} 1 + x - \left( x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 + \dots \right) \\ - \left( 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 + \dots \right) \\ = \int_0^x (x-t)(a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 + a_7t^7 + \dots) dt \end{aligned}$$

Simplify the left side and evaluate the integral on the right side.

$$\begin{aligned} \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 - \frac{1}{40320}x^8 - \frac{1}{362880}x^9 \\ = \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 + \frac{a_3}{20}x^5 + \frac{a_4}{30}x^6 + \frac{a_5}{42}x^7 + \frac{a_6}{56}x^8 + \frac{a_7}{72}x^9 + \dots \end{aligned}$$

Match the coefficients of the respective powers of  $x$  to determine  $a_i$ .

$$\begin{array}{lll} \frac{a_0}{2} = \frac{1}{2} & \frac{a_3}{20} = -\frac{1}{120} & \frac{a_6}{56} = -\frac{1}{40320} \quad \dots \\ \frac{a_1}{6} = \frac{1}{6} & \frac{a_4}{30} = \frac{1}{720} & \frac{a_7}{72} = -\frac{1}{362880} \quad \dots \\ \frac{a_2}{12} = -\frac{1}{24} & \frac{a_5}{42} = \frac{1}{5040} & \dots \end{array}$$

So then

$$\begin{array}{lll} a_0 = 1 & a_3 = -\frac{1}{6} & a_6 = -\frac{1}{720} \quad \dots \\ a_1 = 1 & a_4 = \frac{1}{24} & a_7 = -\frac{1}{5040} \quad \dots \\ a_2 = -\frac{1}{2} & a_5 = \frac{1}{120} & \dots \end{array}$$

and

$$\begin{aligned}u(x) &= 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 - \frac{1}{720}x^6 - \frac{1}{5040}x^7 + \dots \\&= \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots\right) + \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots\right) \\&= \sin x + \cos x.\end{aligned}$$