Exercise 11

Use the *Laplace transform method* to solve the Volterra integral equations of the first kind:

$$3 - 7x + x^2 + \sinh x - 3\cosh x = \int_0^x (x - t - 3)u(t)\,dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)\,dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L}\left\{\int_0^x f(x-t)g(t)\,dt\right\}$$

Take the Laplace transform of both sides of the integral equation.

$$\mathcal{L}\{3 - 7x + x^2 + \sinh x - 3\cosh x\} = \mathcal{L}\left\{\int_0^x (x - t - 3)u(t)\,dt\right\}$$

Use the fact that the Laplace transform is linear on the left side and apply the convolution theorem on the right side.

$$\begin{aligned} 3\mathcal{L}\{1\} - 7\mathcal{L}\{x\} + \mathcal{L}\{x^2\} + \mathcal{L}\{\sinh x\} - 3\mathcal{L}\{\cosh x\} &= \mathcal{L}\{x - 3\}U(s) \\ 3\left(\frac{1}{s}\right) - 7\left(\frac{1}{s^2}\right) + \frac{2}{s^3} + \frac{1}{s^2-1} - 3\left(\frac{s}{s^2-1}\right) &= (\mathcal{L}\{x\} - 3\mathcal{L}\{1\})U(s) \\ \frac{3}{s} - \frac{7}{s^2} + \frac{2}{s^3} + \frac{1-3s}{s^2-1} &= \left(\frac{1}{s^2} - \frac{3}{s}\right)U(s) \end{aligned}$$

Solve for $U(s)$.

$$\begin{aligned} (1-3s)U(s) &= 3s - 7 + \frac{2}{s} + \frac{s^2 - 3s^3}{s^2-1} \\ &= \frac{3s^2 - 7s + 2}{s} + \frac{s^2(1-3s)}{s^2-1} \\ &= \frac{(2-s)(1-3s)}{s} + \frac{s^2(1-3s)}{s^2-1} \\ U(s) &= \frac{2-s}{s} + \frac{s^2}{s^2-1} \\ &= \frac{2}{s} + \frac{1}{s^2-1} \end{aligned}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$\begin{aligned} u(x) &= \mathcal{L}^{-1}\{U(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{2}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} \\ &= 2 + \sinh x \end{aligned}$$