Exercise 2

Use the *Laplace transform method* to solve the Volterra integral equations of the first kind:

$$e^x + \sin x - \cos x = \int_0^x 2e^{x-t}u(t)\,dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx}f(x)\,dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L}\left\{\int_0^x f(x-t)g(t)\,dt\right\}$$

Take the Laplace transform of both sides of the integral equation.

$$\mathcal{L}\{e^x + \sin x - \cos x\} = \mathcal{L}\left\{\int_0^x 2e^{x-t}u(t)\,dt\right\}$$

Use the fact that the Laplace transform is linear on the left side and apply the convolution theorem on the right side.

$$\mathcal{L}\{e^x\} + \mathcal{L}\{\sin x\} - \mathcal{L}\{\cos x\} = \mathcal{L}\{2e^x\}U(s)$$

$$\frac{1}{s-1} + \frac{1}{s^2+1} - \frac{s}{s^2+1} = \frac{2}{s-1}U(s)$$

Solve for $U(s)$.

$$\begin{aligned}\frac{2}{s-1}U(s) &= \frac{1}{s-1} - \frac{s-1}{s^2+1}\\ 2U(s) &= 1 - \frac{(s-1)^2}{s^2+1}\\ &= 1 - \frac{s^2-2s+1}{s^2+1}\\ &= \frac{2s}{s^2+1}\end{aligned}$$

So then

$$U(s) = \frac{s}{s^2+1}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$\begin{aligned}u(x) &= \mathcal{L}^{-1}\{U(s)\}\\ &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}\\ &= \cos x\end{aligned}$$